Soliton evolution and radiation loss for the sine-Gordon equation

Noel F. Smyth*
Department of Mathematics and Statistics, The King's Buildings, University of Edinburgh, Edinburgh, Scotland, United Kingdom, EH9 3JZ

Annette L. Worthy
School of Mathematics and Applied Statistics, University of Wollongong, Northfields Avenue, Wollongong, New South Wales 2522, Australia

(Received 20 January 1999)

An approximate method for describing the evolution of solitonlike initial conditions to solitons for the sine-Gordon equation is developed. This method is based on using a solitonlike pulse with variable parameters in an averaged Lagrangian for the sine-Gordon equation. This averaged Lagrangian is then used to determine ordinary differential equations governing the evolution of the pulse parameters. The pulse evolves to a steady soliton by shedding dispersive radiation. The effect of this radiation is determined by examining the linearized sine-Gordon equation and loss terms are added to the variational equations derived from the averaged Lagrangian by using the momentum and energy conservation equations for the sine-Gordon equation. Solutions of the resulting approximate equations, which include loss, are found to be in good agreement with full numerical solutions of the sine-Gordon equation. [S1063-651X(99)10508-7]

PACS number(s): 41.20.Jb, 52.35.Mw, 02.60.Lj

I. INTRODUCTION

The sine-Gordon equation

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}+\sin u=0 \tag{1}$$

is one of a number of equations describing nonlinear wave motion, which can be solved by the inverse scattering method [1]. This equation arises in a diverse range of areas of physics, for example, crystal dislocation theory [1], self-induced transparency [1], laser physics [1], and particle physics [2]. A related equation, the sinh-Gordon equation, for which the $\sin u$ term is replaced by $\sinh u$, arises in black-hole theory in connection with Hawking radiation [3,4]. The inverse scattering solution shows that any initial condition (with suitably bounded derivative at infinity) will evolve into a finite number of soliton solutions plus dispersive radiation. The soliton solutions of the sine-Gordon equation are

$$u=\pm 4\tan^{-1}\exp\left(\pm\frac{x-Ut}{\sqrt{1-U^2}}\right). \tag{2}$$

Inverse scattering gives that the solitons formed from a given initial condition are determined by the discrete spectrum of a linear eigenvalue problem, and so are, in principle, easily calculated. However, the dispersive radiation shed as the initial condition evolves is given by the solution of a linear integral equation, this solution being nontrivial. Therefore, while it is straightforward to determine the final steady state for a given initial condition, the actual time evolution to this steady state is difficult to determine.

In the present paper an alternative to using the inverse scattering method to describe the evolution of an initial condition for the sine-Gordon equation into solitons plus dispersive radiation will be developed. This approximate method is based on using a trial function in the Lagrangian for the sine-Gordon equation. The effect of the shed dispersive radiation on the evolving soliton is determined by an appropriate solution of the linearized sine-Gordon (Klein-Gordon) equation and the momentum and energy conservation equations for the sine-Gordon equation. A similar approach has been found to be successful for the nonlinear Schrödinger equation [5], which also has an inverse scattering solution [6]. Momentum and energy conservation equations have been used to derive approximate ordinary differential equations describing pulse evolution for the Korteweg–de Vries [7] and Kadomtsev-Petviashvili (KP) [8] equations, both of which also have inverse scattering solutions [6]. One advantage of using approximate methods to derive equations describing pulse evolution is that they can be extended to equations that do not possess an inverse scattering solution, for example, the mKdV equation [9] and coupled nonlinear Schrödinger equations [10].

The Lagrangian method for deriving approximate ordinary differential equations describing pulse evolution for the sine-Gordon equation, which is developed in the present paper, has similarities to that for the nonlinear Schrödinger equation [5]. This is not unexpected as the inverse scattering solutions for the sine-Gordon and nonlinear Schrödinger equations are similar [1]. Solutions of the approximate equations are compared with full numerical solutions of the sine-Gordon equation and good agreement is found. An advantage of the approximate method developed in the present paper is that it can be extended to analyze pulse evolution for perturbed sine-Gordon equations that do not possess an inverse scattering solution, such as those arising in particle physics [2]. Such extensions will be the subject of future work.

*Electronic address: noel@maths.ed.ac.uk

II. APPROXIMATE EQUATIONS

In the present paper the evolution of the initial condition

$$u=4\tan^{-1}\exp\left(-\frac{x-U_0t}{w_0}\right) \tag{3}$$

will be considered since it represents a simple initial condition for which the calculations involved in obtaining the approximate equations are straightforward. The relation between this initial condition and soliton solution (2) is easily seen.

The sine-Gordon equation (1) has the Lagrangian

$$L=\tfrac{1}{2}u_t^2-\tfrac{1}{2}u_x^2+\cos u. \tag{4}$$

An application of Nöther's theorem [11] using this Lagrangian shows that sine-Gordon equation (1) has the momentum conservation equation

$$\frac{\partial}{\partial t}(u_tu_x)-\frac{\partial}{\partial x}\left(\frac{1}{2}u_t^2+\frac{1}{2}u_x^2+\cos u\right)=0 \tag{5}$$

and the energy conservation equation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}u_t^2+\frac{1}{2}u_x^2-\cos u\right)-\frac{\partial}{\partial x}(u_tu_x)=0. \tag{6}$$

To obtain approximate equations describing the evolution of initial condition (3) we assume the form

$$u=4\tan^{-1}\exp\left(-\frac{x-\xi(t)}{w(t)}\right) \tag{7}$$

for the time evolution of u. This type of approximate solution is widely used in field theory, where it is called a collective coordinates solution (a standard reference is Rajaraman [12]). It can be seen that this approximate solution is a varying solitonlike pulse, which can evolve from initial condition (3) to steady soliton (2). The velocity of the pulse is $U=\xi'(t)$. From initial condition (3) it can be seen that $w(0)=w_0$ and $U(0)=U_0$. The derivative u_x of assumed form (7) has a pulselike shape with amplitude $a=-2/w$.

The approximate equations for the pulse parameters $w(t)$ and $U(t)$ are obtained from variations of the averaged Lagrangian

$$\mathcal{L}=\int_{-\infty}^{\infty}L\,dx. \tag{8}$$

By substituting approximate solution (7) into Lagrangian (4) it is found that the averaged Lagrangian is

$$\mathcal{L}=\frac{\pi^2}{3}\frac{w'^2}{w}+4\frac{U^2}{w}-\frac{4}{w}-4w. \tag{9}$$

Taking variations of this averaged Lagrangian with respect to w and ξ gives the ordinary differential equations governing the pulse parameters as

$$\delta w:\quad \frac{2\pi^2}{3w}\frac{d^2w}{dt^2}-\frac{\pi^2}{3w^2}\left(\frac{dw}{dt}\right)^2+4\frac{U^2}{w^2}-\frac{4}{w^2}+4=0, \tag{10}$$

$$\delta\xi:\quad \frac{d}{dt}\left(\frac{U}{w}\right)=0. \tag{11}$$

The velocity equation shows that $U/w=U_0/w_0$. Variational equation (10) for the width w then has the fixed point

$$w_f=\frac{1}{\sqrt{1+\left(\frac{U_0}{w_0}\right)^2}}, \tag{12}$$

so that the fixed point for the velocity is

$$U_f=\frac{U_0}{w_0\sqrt{1+\left(\frac{U_0}{w_0}\right)^2}}. \tag{13}$$

This fixed point is soliton solution (2). However, since variational equations (10) and (11) do not contain any damping terms, the pulse cannot evolve to this steady state. The extension of these equations to include the effect of the dispersive radiation shed as the pulse evolves is considered in the next section.

Momentum and energy conservation equations (5) and (6) can also be averaged to give the momentum and energy conservation integrals

$$\frac{d}{dt}\int_{-\infty}^{\infty}u_tu_x\,dx=0 \tag{14}$$

and

$$\frac{d}{dt}\int_{-\infty}^{\infty}\left(\frac{1}{2}u_t^2+\frac{1}{2}u_x^2-\cos u\right)dx=0, \tag{15}$$

respectively. Substituting trial solution (7) into the momentum conservation integral yields variational equation (11), while the energy conservation integral yields the energy conservation equation

$$\frac{d}{dt}\left(\frac{\pi^2}{3}\frac{w'^2}{w}+\frac{4U^2}{w}+\frac{4}{w}+4w\right)=0, \tag{16}$$

which can be reduced to variational equation (10). It is, therefore, apparent that the variational equations yield the equations for momentum and energy conservation. Energy conservation equation (16) can be integrated to give the exact solution for the width of the pulse as

$$w=\frac{w_0(U_0^2+1+w_0^2)}{2(w_0^2+U_0^2)}+\frac{w_0(U_0^2-1+w_0^2)}{2(w_0^2+U_0^2)}\times\cos\left(\frac{2\sqrt{3}}{\pi}\sqrt{1+\frac{U_0^2}{w_0^2}}\,t\right). \tag{17}$$

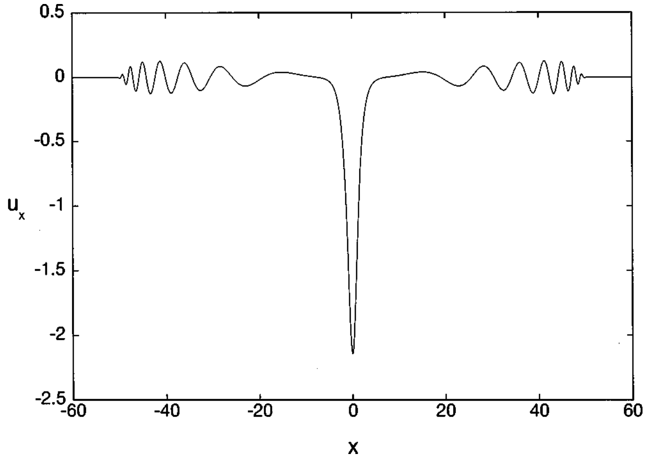

FIG. 1. Full numerical solution for u_x at $t=50$ for initial condition (3) with $w_0=0.6$ and $U_0=0$.

The width w of the pulse and the amplitude $a=-2/w$ of the derivative u_x then oscillate about a mean value. Similar oscillatory behavior also occurs for evolving pulses for the nonlinear Schrödinger equation [5]. Full numerical solutions of the sine-Gordon equation show that the pulse oscillates to a steady state [see Fig. 2(b)]. To enable the approximate solution to approach a steady state the effect of the dispersive radiation shed as the pulse evolves must be included, this being the subject of the next section.

III. DISPERSIVE RADIATION

Figure 1 shows the full numerical solution of sine-Gordon equation (1) for u_x at $t=50$ for the initial condition (3) with $w_0=0.6$ and $U_0=0$. It can be seen that dispersive radiation of small amplitude is shed by the evolving pulse. Now far ahead of the pulse, $u\to 0$ and far behind the pulse, $u\to 2\pi$. Therefore, the shed dispersive radiation is governed by the linearized sine-Gordon equation

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}+u=0, \tag{18}$$

which is the Klein-Gordon equation [13]. This equation is hyperbolic with characteristic velocities ± 1. The resulting wave fronts at $x=\pm t$ can be clearly seen in Fig. 1.

Let us consider the case of a pulse with zero velocity $U=0$ first. It can be seen from Fig. 1 that the radiation in the vicinity of the pulse u_x is flat. This is to be expected as the group velocity $c_g=k/\sqrt{1+k^2}$ for the Klein-Gordon equation shows that waves of low wave number have low group velocity. The radiation in the vicinity of the pulse can then be approximated by $u=g(t)x$, where g is to be determined. On noting that there is no radiation at $t=0$, Klein-Gordon equation (18) can be solved using Laplace transforms to give

$$u=-\int_x^t J_0(\sqrt{\tau^2-x^2})g(t-\tau)d\tau \tag{19}$$

for $x\geqslant 0$ and a symmetric solution for $x\leqslant 0$. Here $J_0(x)$ is the Bessel function of order zero. In obtaining this solution the edge of the radiation has been set at $x=0$ for simplicity, so that $u_x(0,t)=g(t)$. Since the dispersive radiation spreads as t increases, this is a valid approximation for large time. The effect of the dispersive radiation on the evolution of the pulse can now be found from this solution and the momentum and energy conservation equations. Differentiating linear solution (19) gives

$$\begin{aligned} u_t&=-\int_x^t J_0(\sqrt{\tau^2-x^2})g'(t-\tau)d\tau \\ &=-g(t-x)+\int_x^t \frac{\tau}{\sqrt{\tau^2-x^2}}J_1(\sqrt{\tau^2-x^2})g(t-\tau)d\tau, \end{aligned} \tag{20}$$

$$u_x=g(t-x)-\int_x^t \frac{x}{\sqrt{\tau^2-x^2}}J_1(\sqrt{\tau^2-x^2})g(t-\tau)d\tau, \tag{21}$$

on noting that $g(0)=0$ as there is no radiation initially.

The linearized forms of momentum and energy conservation equations (5) and (6) for the sine-Gordon equation are

$$\frac{\partial}{\partial t}(u_t u_x)-\frac{1}{2}\frac{\partial}{\partial x}(u_t^2+u_x^2-u^2)=0 \tag{22}$$

and

$$\frac{1}{2}\frac{\partial}{\partial t}(u_t^2+u_x^2+u^2)-\frac{\partial}{\partial x}(u_t u_x)=0, \tag{23}$$

respectively. Integrating linearized energy conservation equation (23) from the pulse $x=0$ to the front $x=t$ gives the energy lost to the radiation propagating into $x>0$ as

$$\frac{d}{dt}\frac{1}{2}\int_0^t (u_t^2+u_x^2+u^2)dx=g^2(t)-g(t)\int_0^t J_1(\tau)g(t-\tau)d\tau. \tag{24}$$

Using the symmetry of the solution for the radiation, the total energy flux from the pulse to the dispersive radiation is, therefore,

$$\frac{dH}{dt}=-2g^2(t)+2g(t)\int_0^t J_1(\tau)g(t-\tau)d\tau. \tag{25}$$

Here H refers to the energy density in energy conservation equation (6). To complete the modification of variational equations (10) and (11) to include loss to dispersive radiation, the parameter g is now related to the pulse width w.

Let us expand the energy density H about the fixed point $w=w_f$ ($U_f=0$ in the case under consideration). Setting $w=w_f+w_1$, where $|w_1|$ is small, the energy density in energy equation (6) becomes

$$H=H_f+\delta H=\frac{8}{w_f}+\frac{\pi^2}{3}\frac{(w_1')^2}{w_f}+4\frac{w_1^2}{w_f^3}. \tag{26}$$

Using exact solution (17) w_1 can be replaced by w_1'' to give the perturbed Hamiltonian as

$$\delta H=\frac{\pi^2}{3}\frac{(w_1')^2}{w_f}+\frac{\pi^2}{12}\frac{\pi^2}{3}w_f(w_1'')^2. \tag{27}$$

Note that in replacing w_1 by w_1'', the term

$$\frac{6}{\pi^2}\left(\frac{U_0^2}{w_0}+\frac{1}{w_0}+w_0-\frac{2}{w_f}\right) \tag{28}$$

has been ignored since only the derivative of the Hamiltonian matters in the calculation of the energy loss due to the dispersive radiation. The relation between w and g will now be obtained by equating δH to the energy shed to the dispersive radiation. From linearized energy conservation equation (23) the energy in the radiation shed to the right of the pulse is given by

$$\frac{1}{2}\int_0^t(u_t^2+u_x^2+u^2)dx \tag{29}$$

with the radiation u given by Laplace transform solution (19). The integrals in energy expression (29) are difficult to evaluate using this Laplace transform solution, so approximations for large time will now be made. These large time approximations are consistent with expanding the energy about the fixed point.

For large time, solution (19) for $u(0,t)$ is

$$u(0,t)\sim-\int_0^\infty J_0(\tau)g(t-\tau)d\tau. \tag{30}$$

If g were a constant, this could be further reduced to

$$u(0,t)\sim-g\int_0^\infty J_0(\tau)d\tau=-g, \tag{31}$$

on noting that the integral from $x=0$ to $x=\infty$ of $J_0(x)$ is 1 [14]. Now it will be found that g is a slowly decaying oscillatory function of t. Therefore, Eq. (31) is a valid approximation for $u(0,t)$ for large t if g is taken as the slowly varying mean value of these oscillations. In a similar manner, expression (20) for $u_t(0,t)$ can be approximated for large t by

$$u_t(0,t)=-g'. \tag{32}$$

Integral (29) for the energy in the radiation shed to the right of the pulse is now approximated by the trapezoidal rule using Eq. (21) for $u_x(0,t)$, Eq. (31) for $u(0,t)$ and Eq. (32) for $u_t(0,t)$, noting that u, u_x, and u_t are all zero at the front $x=t$. Symmetry then finally gives that the total energy in the dispersive radiation is given by the approximation

$$H_r=\frac{1}{2}\int_{-t}^t(u_t^2+u_x^2+u^2)dx=g^2t+\frac{1}{2}g'^2t \tag{33}$$

for large t.

Comparing expression (33) for the energy in the radiation to expression (27) for the perturbed Hamiltonian of the pulse near the fixed point, it can be seen that we can equate

$$g^2=\frac{\pi^2}{3}\frac{(w_1')^2}{w_ft}\sim\frac{\pi^2}{3}\frac{(w')^2}{wt} \tag{34}$$

and

$$g'^2=\frac{\pi^2w_f}{6}\frac{\pi^2}{3}\frac{(w_1'')^2}{t}\sim\frac{\pi^2}{6}\frac{\pi^2}{3}\frac{(w'')^2}{wt}. \tag{35}$$

These expressions for g and g' do not quite agree, but are in approximate agreement as $\pi/\sqrt{6}=1.28\ldots\sim1$ (note that $w_f=1$ for $U_0=0$). A similar slight disagreement in the dispersive radiation term was found for the nonlinear Schrödinger equation [5].

The preceding derivation of the energy lost to the dispersive radiation was for the case $U=0$. When $U\neq0$ the radiation is given by the solution of a moving boundary problem for Klein-Gordon equation (18), the moving boundary being at the pulse position $x=\xi$. This moving boundary problem is difficult to solve. However if $U\neq0$ were a constant, then by the Galilean invariance of the sine-Gordon equation the energy loss to the radiation would still be given by Eq. (25). It can be further shown from momentum equation (22) for the Klein-Gordon equation that for U taken as a constant, the momentum loss to the radiation is given by

$$\begin{aligned}\frac{dP}{dt}&=\frac{d}{dt}\int_{-t}^t u_tu_xdx\\&=-2U\left[g^2(t)-g(t)\int_0^tJ_1(\tau)g(t-\tau)d\tau\right].\end{aligned} \tag{36}$$

If U were slowly varying, then this expression would give the momentum loss to the dispersive radiation. It will be found in the next section from full numerical solutions of the sine-Gordon equation that for large time U does not vary greatly. Therefore, we shall use this expression for the momentum loss when U is not constant.

Adding energy-loss expression (25) to variational equation (10) for w and momentum-loss expression (36) to variational equation (11) for U, we finally have that the equations, including momentum and energy loss, governing the evolution of the pulse are

$$\begin{aligned}&\frac{2\pi^2}{3w}\frac{d^2w}{dt^2}-\frac{\pi^2}{3w^2}\left(\frac{dw}{dt}\right)^2+4\frac{U^2}{w^2}-\frac{4}{w^2}+4\\&\quad=-\frac{2\pi^2}{3wt}\frac{dw}{dt}+\frac{2\pi^2}{3\sqrt{wt}}\int_0^tJ_1(t-\tau)\frac{w'(\tau)}{\sqrt{\tau w(\tau)}}d\tau\end{aligned} \tag{37}$$

and

$$\frac{d}{dt}\left(\frac{U}{w}\right)=\frac{\pi^2U}{12}\frac{dw}{dt}\left[\frac{1}{wt}\frac{dw}{dt}-\frac{1}{\sqrt{wt}}\int_0^tJ_1(t-\tau)\frac{w'(\tau)}{\sqrt{\tau w(\tau)}}d\tau\right]. \tag{38}$$

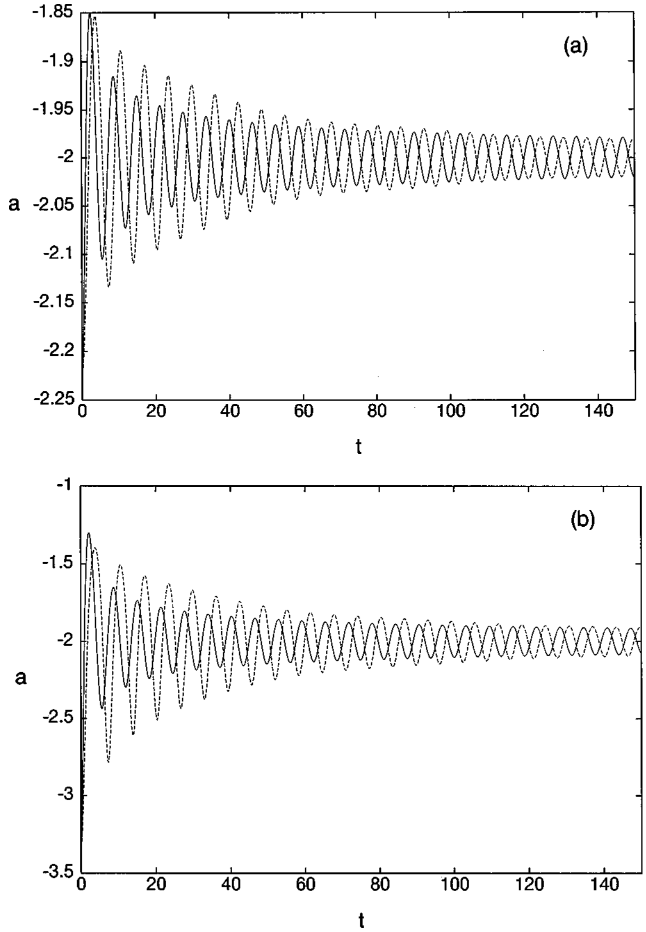

FIG. 2. Amplitude a of u_x as a function of t for initial condition (3). Full numerical solution, —; solution of approximate equations, – – – . (a) $w_0 = 0.9$ and $U_0 = 0$ and (b) $w_0 = 0.6$ and $U_0 = 0$.

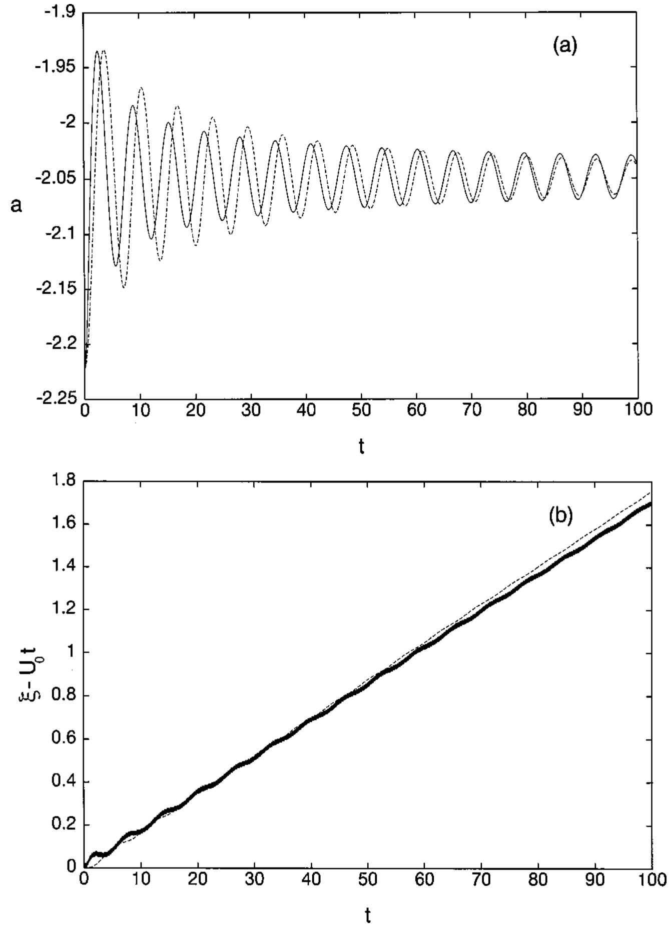

FIG. 3. Pulse evolution for initial condition (3) with $w_0 = 0.9$ and $U_0 = 0.2$. Full numerical solution, —; solution of approximate equations, – – – . (a) Amplitude a of u_x and (b) position ξ of pulse minus position of pulse with velocity U_0.

In the next section solutions of these approximate equations will be compared with full numerical solutions of sine-Gordon equation (1). The approximate equations were solved numerically using a fourth-order Runge-Kutta scheme with the integrals on the right-hand side of the equations evaluated using the trapezoidal rule.

IV. COMPARISON WITH NUMERICAL SOLUTIONS

Sine-Gordon equation (1) was solved numerically using second-order centered differences in space and time. This scheme was tested by propagating soliton solution (2), which was found to propagate without change of form to within less than a percent error. The solutions obtained from this numerical scheme will now be compared with numerical solutions of approximate equations (37) and (38).

In Fig. 2 the amplitude a of u_x as given by the solution of the approximate equations and by the full numerical solution of the sine-Gordon equation is shown for the initial conditions $w_0 = 0.9$ and $w_0 = 0.6$ with $U_0 = 0$, so that $U = 0$. Figure 2(a) shows the amplitude for the initial condition near the soliton solution and Fig. 2(b) shows the amplitude for the initial condition far from the steady soliton. It can be seen that the agreement between the two solutions is excellent for both initial conditions, with the major disagreement being a constant phase difference. It is noted that the approximate method does not yield the phase of the solution. The decay rate of the approximate solution is in excellent agreement with the numerical decay rate for large time. The agreement for small times is not so good, but this is expected from the derivation of the radiation loss for the approximate equations. Since the radiation loss terms were derived under the assumption that $U = 0$, the good agreement shown in the figures is to be expected.

Figure 3 shows amplitude and position comparisons for the initial conditions $w_0 = 0.9$ and $U_0 = 0.2$. It can again be seen that there is good agreement between the approximate and numerical solutions. The period of the approximate solution is slightly smaller than the numerical period and the decay rate is slightly larger. The agreement between the peak position ξ as given by the two solutions is also good. Good agreement between the approximate and numerical solutions continues up to $U_0 \sim 0.4$. Figure 4 shows amplitude and position comparisons for the initial conditions $w_0 = 0.8$ and $U_0 = 0.5$. While the agreement between the positions is good, the agreement between the amplitudes is only fair. The radiation loss terms in the approximate equations were derived under the assumption that U was a constant. It is clear that as U_0 increases this assumption becomes less valid. It can be seen from Fig. 4(a) that the amplitude oscillations as given by the approximate equations have an anharmonic compo-

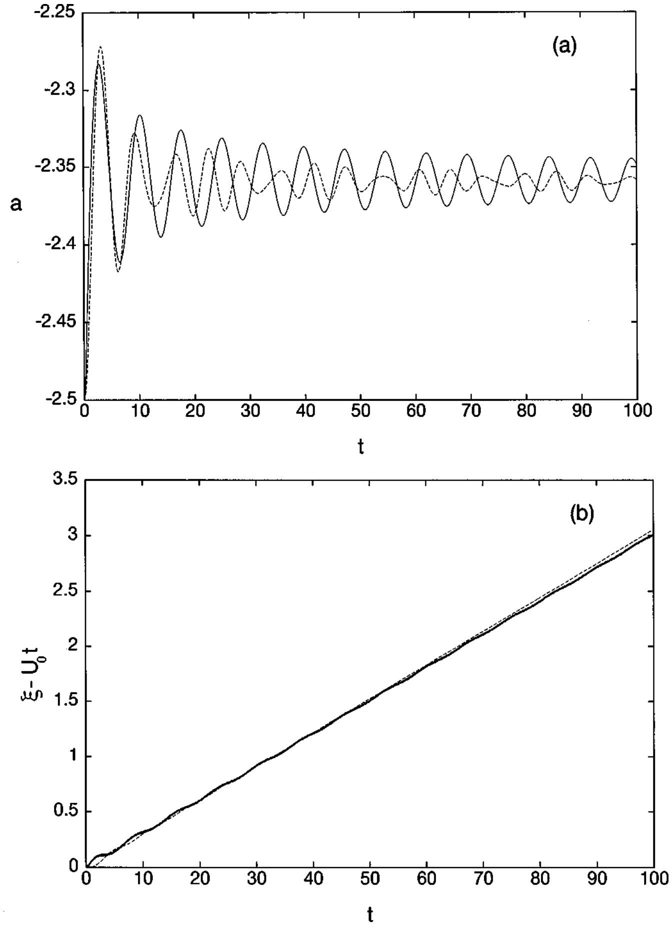

FIG. 4. Pulse evolution for initial condition (3) with $w_0=0.8$ and $U_0=0.5$. Full numerical solution, —; solution of approximate equations, – – – . (a) Amplitude a of u_x and (b) position ξ of pulse minus position of pulse with velocity U_0.

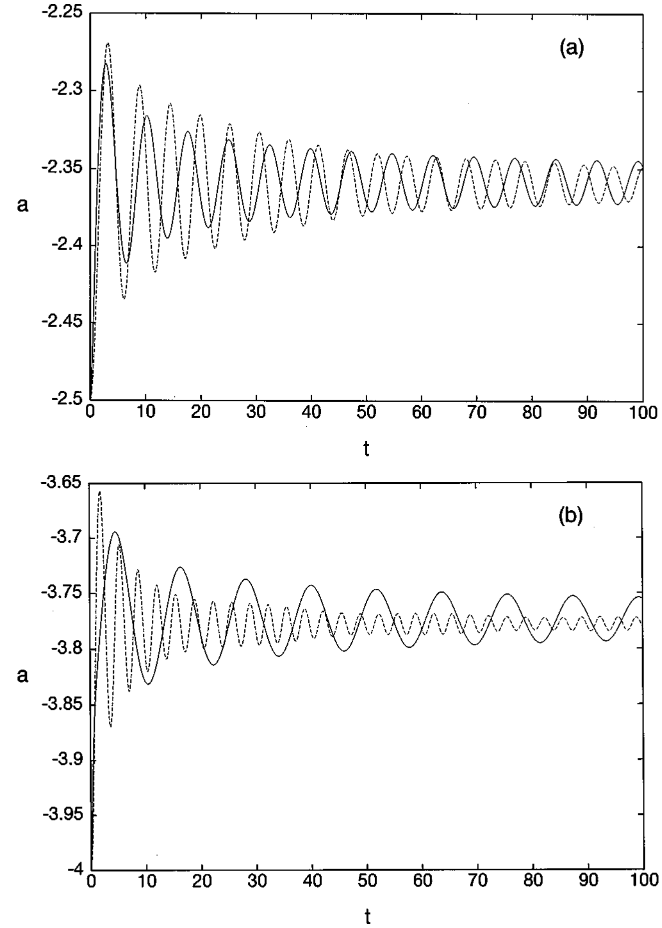

FIG. 5. Pulse evolution using approximate equations (39) and (40). Full numerical solution, —; solution of approximate equations, – – – . (a) Amplitude a of u_x for $w_0=0.8$ and $U_0=0.5$ and (b) amplitude a of u_x for $w_0=0.5$ and $U_0=0.8$.

nent. This is due to a mismatch between the periods of J_1 and w in the integral term of the radiation damping terms in the approximate equations. For large x, $J_1(x)$ has period $2\pi\sim 6.283\ldots$ [14]. For $U=0$ exact solution (17) has period $\pi^2/\sqrt{3}\approx 5.698$, so that J_1 and w have nearly the same period. However, as U_0 increases it can be seen from exact solution (17) that the period of w becomes shorter than the period of J_1. This suggests that to correct the anharmonic behavior the argument of J_1 should be adjusted to make the periods of J_1 and w similar for large t. Upon doing this approximate equations (37) and (38) become

$$\frac{2\pi^2}{3w}\frac{d^2w}{dt^2}-\frac{\pi^2}{3w^2}\left(\frac{dw}{dt}\right)^2+4\frac{U^2}{w^2}-\frac{4}{w^2}+4$$
$$=-\frac{2\pi^2}{3wt}\frac{dw}{dt}+\frac{2\pi^2}{3\sqrt{wt}}\int_0^t$$
$$\times J_1(\sqrt{1+(U_0/w_0)^2}\,(t-\tau))\frac{w'(\tau)}{\sqrt{\tau w(\tau)}}d\tau \qquad (39)$$

and

$$\frac{d}{dt}\left(\frac{U}{w}\right)=\frac{\pi^2U}{12}\frac{dw}{dt}\left[\frac{1}{wt}\frac{dw}{dt}-\frac{1}{\sqrt{wt}}\right.$$
$$\left.\times\int_0^t J_1(\sqrt{1+(U_0/w_0)^2}\,(t-\tau))\frac{w'(\tau)}{\sqrt{\tau w(\tau)}}d\tau\right]. \qquad (40)$$

Since the radiation loss terms were derived under the assumption that $U_0=0$, these equations are as justified as Eqs. (37) and (38) for $U_0\neq 0$.

Figure 5 shows comparisons between the full numerical solution of the sine-Gordon equation and the solution of frequency adjusted equations (39) and (40) for the amplitude a of u_x. It can be seen that the anharmonic component of the approximate amplitude oscillation has now disappeared. Figure 5(a) shows the comparison for the initial parameter values $w_0=0.8$ and $U_0=0.5$, as for Fig. 4(a). The comparison between the numerical and approximate solutions is now reasonable, with the damping of the approximate solution being slightly stronger than that of the numerical solution and the period of the approximate solution being somewhat shorter. Figure 5(b) shows the same comparison for the initial values $w_0=0.5$ and $U_0=0.8$. The period of the approximate solution is now significantly shorter than the numerical period and the decay significantly stronger. However, the agreement between the final steady states is good.

V. CONCLUSIONS

The evolution of solitonlike initial conditions to soliton solutions for the sine-Gordon equation has been examined. It has been shown that in order to obtain good agreement with full numerical solutions, the effect of the dispersive radiation shed as the pulse evolves must be included. The effect of this dispersive radiation was found by using a suitable solution of the linearized sine-Gordon equation (the Klein-Gordon equation) in conjunction with the momentum and energy conservation equations for the sine-Gordon equation. While the exact inverse scattering solution of the sine-Gordon equation provides this information in principle, in practice it is difficult to explicitly obtain it. Furthermore, the approximate method outlined in the present paper can be extended to sine-Gordon-type equations for which there are no inverse scattering solutions. These extensions will be the subject of further work.

[1] G.L. Lamb, *Elements of Soliton Theory* (Wiley Interscience, New York, 1980).

[2] W.J. Zakrzewski, *Low Dimensional Sigma Models* (Hilger, Bristol, 1989).

[3] R. Jackiw and C. Teitelboim, in *Quantum Theory of Gravity*, edited by S.M. Christensen (Hilger, Bristol, 1984).

[4] J. Gegenberg and G. Kunstatter, Phys. Lett. B **413**, 274 (1997).

[5] W.L. Kath and N.F. Smyth, Phys. Rev. E **51**, 1484 (1995).

[6] A.C. Newell, *Solitons in Mathematics and Physics* (SIAM, Philadelphia, 1985).

[7] W.L. Kath and N.F. Smyth, Phys. Rev. E **51**, 661 (1995).

[8] A.A. Minzoni and N.F. Smyth, Wave Motion **24**, 291 (1996).

[9] N.F. Smyth and A.L. Worthy, Wave Motion **21**, 263 (1995).

[10] N.F. Smyth and A.L. Worthy, J. Opt. Soc. Am. B **14**, 2610 (1997).

[11] I.M. Gelfand and S.V. Fomin, *Calculus of Variations* (Prentice-Hall, Englewood Cliffs, NJ, 1963).

[12] R. Rajaraman, *Solitons and Instantons: An Introduction to Solitons and Instantons in Quantum Field Theory* (North-Holland, Amsterdam, 1982).

[13] G.B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1974).

[14] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I.A. Stegun (Dover Publications, New York, 1972).